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Solution by WILLIAM HOOVER, A. M., Ph. D., Professor of Mathematics and Astronomy, Ohio University, Athens, O.

Let  $l$  = the entire length of the chain,  $a$  = the radius of the wheel,  $x$  = the longest part hanging down,  $l - (\pi a + x)$  = the shortest part,  $\theta$  = the angle which the radius through any point of the string makes with the vertical diameter and positive in the direction left to right,  $T$  and  $T'$  the tensions at any points in those parts of the string to which  $x$  and  $l - (\pi a + x)$  belong, and  $\mu$  = the coefficient of friction.

We now have, from the usual theory, a unit of length of the chain being taken as a unit of weight.

$$T = Ce^{\mu\theta} + \frac{a}{1+\mu^2} [2\mu\sin\theta + (1-\mu^2)\cos\theta] \dots (1).$$

When  $\theta = \frac{1}{2}\pi$ ,  $T = x$ , and (1) is

$$x = Ca^{\frac{1}{2}(\mu\pi)} + \frac{2\mu a}{1+\mu^2}, \text{ or } C = \left(x - \frac{2\mu a}{1+\mu^2}\right) e^{-\frac{1}{2}(\mu\pi)} \dots (2),$$

and (1) is  $T = \left(x - \frac{2\mu a}{1+\mu^2}\right) e^{\mu(\theta - \frac{1}{2}\pi)} + \frac{a}{1+\mu^2} [-2\mu\sin\theta + (1-\mu^2)\cos\theta] \dots (3).$

In like manner,  $T' = l - (\pi a + x)$  when  $\theta = -\frac{1}{2}\pi$ , and

$$T' = \left(l - \pi a - x + \frac{2\mu a}{1+\mu^2}\right) e^{\mu(\theta + \frac{1}{2}\pi)} + \frac{a}{1+\mu^2} [-2\mu\sin\theta + (1-\mu^2)\cos\theta] \dots (4).$$

For equilibrium,  $T' = T$  at the vertex, where  $\theta = 0$ .

$$\therefore \left(l - \pi a - x + \frac{2\mu a}{1+\mu^2}\right) e^{\frac{1}{2}(\mu\pi)} = \left(x - \frac{2\mu a}{1+\mu^2}\right) e^{-\frac{1}{2}(\mu\pi)} \dots (5),$$

giving  $x = \frac{l - \pi a}{1 + e^{-\mu\pi}} + \frac{2\mu a}{1 + \mu^2} \dots (6).$

Then  $2x + \pi a - l$  is found, the required length.

Also solved by G. B. M. ZERR.

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### AVERAGE AND PROBABILITY.

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101. Proposed by L. C. WALKER, Assistant Professor of Mathematics, Leland Stanford Jr. University, Palo Alto, Cal.

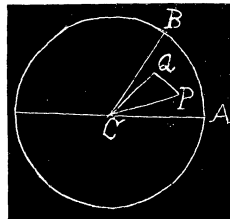
By direct calculation obtain the average distance between two points in the surface of a circle.

I. Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

Let  $P, Q$  be the random points in the sector  $AOB$ .  $AO = r$ ,  $OP = y$ ,  $OQ = x$ ,

$\angle AOB = \beta$ ,  $\angle AOQ = \theta$ ,  $\angle AOP = \varphi$ . An element of the sector at  $P$  is  $ydyd\varphi$ ; at  $Q$ ,  $xdxd\theta$ . The limits of  $x$ , are 0 and  $r$ ; of  $y$ , 0 and  $x$ ; of  $\theta$ , 0 and  $\beta$ ; of  $\varphi$ , 0 and  $\theta$ .

$$PQ = u = [x^2 + y^2 - 2xy\cos(\theta - \varphi)]^{\frac{1}{2}}.$$



$$\therefore \Delta = \frac{\int_0^\beta \int_0^\theta \int_0^r \int_0^x u x dx dy d\theta d\varphi}{\int_0^\beta \int_0^\theta \int_0^r \int_0^x x dx dy d\theta d\varphi}$$

$$\begin{aligned} &= \frac{16}{\beta^2 r^4} \int_0^\beta \int_0^\theta \int_0^r \int_0^x u x y d\theta d\varphi dx dy = \frac{8}{\beta^2 r^4} \int_0^\beta \int_0^\theta \int_0^r \{16\sin^{\frac{1}{2}}(\theta - \varphi) \\ &\quad + 12\sin^{\frac{3}{2}}(\theta - \varphi)\cos(\theta - \varphi) - 2 + 3\cos^2(\theta - \varphi) \\ &\quad + 3\sin^2(\theta - \varphi)\cos(\theta - \varphi)\log[1 + \operatorname{cosec}\tfrac{1}{2}(\theta - \varphi)]\} x^4 d\theta d\varphi dx \\ &= \frac{8r}{15\beta^2} \int_0^\beta \int_0^\theta \{16\sin^{\frac{3}{2}}(\theta - \varphi) + 12\sin^{\frac{5}{2}}(\theta - \varphi)\cos(\theta - \varphi) - 2 + 3\cos^2(\theta - \varphi) \\ &\quad + 4\sin^2(\theta - \varphi)\cos(\theta - \varphi)\log[1 + \operatorname{cosec}\tfrac{1}{2}(\theta - \varphi)]\} d\theta d\varphi \\ &= \frac{4r}{45\beta^2} \int_0^\beta \{48\sin^{\frac{4}{2}}\theta\cos^{\frac{1}{2}}\theta + 12\sin^{\frac{3}{2}}\theta\cos^{\frac{3}{2}}\theta - 32\sin^{\frac{5}{2}}\theta\cos^{\frac{1}{2}}\theta - 6\sin^{\frac{1}{2}}\theta\cos^{\frac{5}{2}}\theta \\ &\quad - 64\cos^{\frac{1}{2}}\theta + 64 + 9\sin\theta\cos\theta + 6\sin^3\theta\log(1 + \operatorname{cosec}\theta)\} d\theta \\ &= \frac{2r}{135\beta^2} \{16\sin^6\tfrac{1}{2}\beta + 16\cos^6\tfrac{1}{2}\beta + 96\sin^5\tfrac{1}{2}\beta + 24\sin^4\tfrac{1}{2}\beta - 12\cos^4\tfrac{1}{2}\beta - 112\sin^3\tfrac{1}{2}\beta \\ &\quad - 36\sin^2\tfrac{1}{2}\beta - 720\sin\tfrac{1}{2}\beta + 27\sin^2\beta + 384\beta - 4 \\ &\quad - 12(\sin^2\beta\cos\beta + 2\cos\beta + 2)\log(1 + \sin\tfrac{1}{2}\beta) + 12(\sin^2\beta\cos\beta \\ &\quad + 2\cos\beta - 2)\log\sin\tfrac{1}{2}\beta\}. \end{aligned}$$

For the circle,  $\beta = 2\pi$ ,  $\Delta = \frac{128r}{45\pi}$ .

For the semicircle,  $\beta = \pi$ ,  $\Delta = \frac{256r}{45\pi} - \frac{1472r}{135\pi^2}$ .

For the quadrant,  $\beta = \frac{1}{2}\pi$ .

$$\Delta = \frac{32r}{135\pi} \left( 48 + \frac{3}{\pi} - \frac{94\sqrt{2}}{\pi} - \frac{6}{\pi} \log \frac{1 + \sqrt{2}}{2} \right).$$

## II. Solution by the PROPOSER.

Take the center,  $C$ , of the circle and the horizontal radius  $CA$  as pole and initial line in polar coördinates.

Let  $P(x, \theta)$  and  $Q(y, \phi)$  be the random points. Then we have

$$PQ = \sqrt{x^2 + y^2 - 2xy \cos(\theta - \phi)}.$$

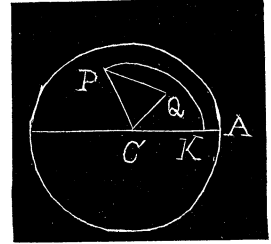
Now with  $C$  as a center and a radius  $CP$  describe the arc  $PK$ .

It is evidently necessary to consider only those positions of the two points in which  $Q$  is confined to the sector  $PCK$ , since under this limitation  $PQ$  will take all possible distances.

An element of area of the circle at the point  $P$  is  $x dx d\theta$ , and at the point  $Q$  it is  $y dy d\phi$ .

The limits of  $y$  are 0 and  $x$ ; of  $x$ , 0 and  $r$ ; of  $\phi$ , 0 and  $\theta$ ; and those of  $\theta$ , 0 and  $2\pi$ .

Consequently, we have for the required mean distance



$$\begin{aligned} M &= \frac{1}{(\pi r^2)^2} \int_0^{2\pi} \int_0^\theta \int_0^r \int_0^x \sqrt{x^2 + y^2 - 2xy \cos(\theta - \phi)} x dx d\theta y dy d\phi \\ &= \frac{4}{3\pi^2 r^4} \int_0^{2\pi} \int_0^\theta \int_0^x \left[ 8\sin^3 \frac{1}{2}(\theta - \phi) + 6\sin^3 \frac{1}{2}(\theta - \phi) \cos(\theta - \phi) \right. \\ &\quad \left. - \frac{3}{2}\sin^2(\theta - \phi) \cos(\theta - \phi) \log \left( \frac{1 + \sin \frac{1}{2}(\theta - \phi)}{\sin \frac{1}{2}(\theta - \phi)} \right) - 1 + \cos^2(\theta - \phi) \right] x^4 d\theta d\phi dx \\ &= \frac{r}{15\pi^2} \int_0^{2\pi} \int_0^\theta \left[ 8\sin \frac{1}{2}(\theta - \phi) + 40\sin \frac{1}{2}(\theta - \phi) \cos^2 \frac{1}{2}(\theta - \phi) \right. \\ &\quad \left. + 6\sin^2(\theta - \phi) \cos(\theta - \phi) \log \left( \frac{1 + \sin \frac{1}{2}(\theta - \phi)}{\sin \frac{1}{2}(\theta - \phi)} \right) \right. \\ &\quad \left. - 48\sin \frac{1}{2}(\theta - \phi) \cos^4 \frac{1}{2}(\theta - \phi) + 3\cos 2(\theta - \phi) - 1 \right] d\theta d\phi \\ &= \frac{2r}{45\pi^2} \int_0^{2\pi} \left[ 32 - 32\cos \frac{1}{2}\theta - 16\sin^2 \frac{1}{2}\theta \cos \frac{1}{2}\theta + 24\sin^4 \frac{1}{2}\theta \cos \frac{1}{2}\theta + 3\sin \theta \cos \theta \right. \\ &\quad \left. + 3\sin^3 \theta \log \left( \frac{1 + \sin \frac{1}{2}\theta}{\sin \frac{1}{2}\theta} \right) \right] d\theta = \frac{128r}{45\pi}. \end{aligned}$$

Professor Walker also furnished a second very excellent solution.

## MISCELLANEOUS.

91. Proposed by ARTEMAS MARTIN, LL. D., U. S. Coast and Geodetic Survey Office, Washington, D. C.

The following sides and area are given for a rational triangle in the table of rational scalene triangles on page 167 of Dr. Halsted's "Metrical Geometry (Boston, 1881), viz.: sides, 21, 61, 65; area, 420. The same sides and area are given in Septimus Tebay's "Measurement" (London and Cambridge, 1868), in a table on page 113.